

## 7. Superfluid

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Learning Aims: You will learn to

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Explain how Bose-Einstein distribution is used to understand Bose-Einstein condensation. Derive the formulae for number of excited bosons and condensation temperature.

Sketch and explain the graph of boson numbers against temperature.

Sketch and explain the graph for heat capacity of helium gas around the condensation temperature.

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Learning Aims: You will learn to

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Sketch and describe the phase diagram of liquid helium-4.

Derive the dispersion relations for a phonon and a real particle in empty space.

State the formula for Landau critical velocity. Describe what it means and the conditions where it is valid.

Describe Landau's dispersion relation for liquid helium. Explain the evidence from heat capacity measurement.

Describe London's explanation for superfluidity.

State the formula for Bose-Einstein distribution. Explain how chemical potential changes with temperature for bosons.

### Superfluid helium

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## Superfluid helium.

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When helium-4 is cooled below 2.17 K, it becomes superfluid. This is a phase of matter that is different from any liquid at room temperature - it has no viscosity at all.

When a stone is dropped in water, it experiences resistance to its movement because of viscosity. When the same stone is dropped in superfluid helium-4, it experiences no resistance at all.

In order to understand these behaviours, we study the following:

1. what zero resistance means for excitations (e.g. phonons),
  2. what the heat capacity can say about excitations,
  3. how a wavefunction may have zero resistance, and
  4. the idea of a macroscopic (giant) wavefunction.
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## Superfluid ideas.

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One simple way to start to understand superfluid is to think about excitations.

The excitation here refers to exciting the fluid, or particles in the fluid, to a higher energy state. Imagine a large body moving through a liquid. It collides with the particles in the fluid.

If the collision is too weak, it may not cause any excitation, since energy levels occur in discrete steps (i.e. they are quantised).

This suggests that there must be a velocity of the body below which there would be no excitation.

No excitation means no viscosity. Why?

If there is resistance, the body must give up some energy. By energy conservation, this can only go to create excitations.

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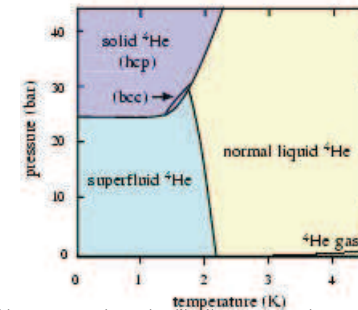
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## Liquid helium-4

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When temperature falls, we expect matter to change from gas to liquid to solid. Helium-4, however, remains a liquid right down to 0 K.

The phase diagram shows that helium can only become a solid if the pressure is higher than 25 atm.



[http://ltdl.tkk.fi/wiki/LT/%C2%B5KI\\_Group/Helium\\_Crystals](http://ltdl.tkk.fi/wiki/LT/%C2%B5KI_Group/Helium_Crystals)

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## Landau critical velocity

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To develop the excitation idea further, we use the dispersion relation. This is the relation between energy and momentum of a particle.

For example, the dispersion relation for a particle of mass  $m$  in free space with energy  $E$  and momentum  $p$  is

$$E = \frac{p^2}{2m}.$$

This comes from the more familiar kinetic energy formula

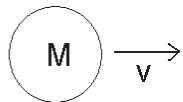
$$E = \frac{1}{2}mv^2$$

when we replace  $v$  by  $p/m$ .

The dispersion relation is a kind of constraint on the possible values of  $E$  and  $p$ .

Return to the idea of a large body moving through a superfluid. We can derive an expression for the velocity that would produce an elementary excitation.

Consider a large body of mass  $M$  moving at velocity  $v$ .



Or, if the particle is a phonon, then the relation is

$$E = pc,$$

where  $c$  here is the speed of sound.

This comes from the more familiar relations

$$v = f\lambda, E = hf \text{ and } p = h/\lambda.$$

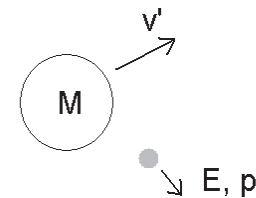
The dispersion relation is again a constraint on the possible values of  $E$  and  $p$ .

To produce a phonon with momentum  $p$ , we must have an energy of exactly  $E = pc$ . If we have more or less than  $E$ , a phonon with a different  $p$  is produced.

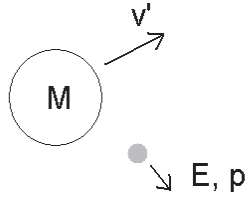
We think of the phonon as an elementary excitation.



Suppose it moves fast enough to produce an elementary excitation (e.g. phonon) of energy  $E$ .



After that, it's velocity changes to  $v'$ .



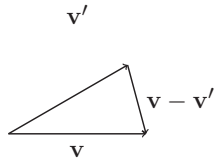
By energy conservation,

$$\frac{1}{2}Mv^2 = \frac{1}{2}Mv'^2 + E.$$

Suppose that the momentum of the excitation is  $\mathbf{p}$ . By momentum conservation,

$$M\mathbf{v} = M\mathbf{v}' + \mathbf{p}.$$

From this vector diagram



we can see that

$$|\mathbf{v} - \mathbf{v}'| \geq v - v',$$

because  $v - v'$  is nearly zero, since  $v' \approx v$ .

Rewrite the equations to give:

$$(v + v')(v - v') = \frac{2E}{M}$$

and

$$\mathbf{v} - \mathbf{v}' = \frac{\mathbf{p}}{M}.$$

From these, we can get

$$2v(v - v') \approx \frac{2E}{M}$$

and

$$|\mathbf{v} - \mathbf{v}'| = \frac{p}{M},$$

where we have used  $v' \approx v$  because the mass of the body is much larger than that of the excitation particle.

Rewriting the equations in this form,

$$(v - v') \approx \frac{E}{vM}$$

and

$$|\mathbf{v} - \mathbf{v}'| = \frac{p}{M},$$

and substituting into

$$|\mathbf{v} - \mathbf{v}'| \geq v - v',$$

we find

$$\frac{p}{M} \geq \frac{E}{vM},$$

or

$$v \geq \frac{E}{p}.$$

So  $v$  must be larger than  $E/p$  in order to produce any excitation.

Recall that  $E$  and  $p$  must be related by the dispersion relation.

Suppose the minimum value of  $E/p$  is not zero. Let this be

$$v_L = \left( \frac{E}{p} \right)_{min}.$$

We have shown that  $v$  must be more than this in order to produce any excitation.

So if the body moves at a velocity below  $v_L$ , it would not produce any excitation. This means it would not experience any viscosity - i.e. the fluid is a superfluid.

$v_L$  is called the Landau critical velocity.

Suppose the range of possible  $E/p$  has a certain minimum value. Let this be

$$v_L = \left( \frac{E}{p} \right)_{min}.$$

So  $v$  of the body must at least be more than this. If not, then  $v$  would be smaller than the smallest  $E/p$ . This means  $v$  is smaller than all possible  $E/p$ . If so, it cannot produce any excitation.

Therefore, if the body moves at a velocity below  $v_L$ , it cannot produce any excitation. This means it would not experience any viscosity, because it cannot lose any energy. Then the fluid is a superfluid.

$v_L$  is called the Landau critical velocity.

Assume that the body loses only a very small fraction of its kinetic energy to the excitation energy. We have shown that

$$v \geq \frac{E}{p}.$$

In words, the ratio of the excitation energy to its momentum cannot be bigger than the velocity of the body.

So  $v$  of the body must be larger than  $E/p$  of the excitation in order to produce any excitation at all.

Recall that the excitation would obey a dispersion relation. E.g. for a phonon,  $E = pc$ , so the ratio of  $E/p = c$ , a fixed number. For other types of excitation, this could be a range of numbers.

Therefore, if the smallest  $E/p$  in a fluid is not zero, then superfluidity is possible. This is because if the body moves at a velocity below  $(E/p)_{min}$ , it experiences no viscosity.

Conversely, if the smallest  $E/p$  in a fluid is zero, then it is not a superfluid. This is because as soon as the body moves, it creates excitation and loses energy. As a result, it experiences viscosity.

Let us now look at the typical dispersion relations in matter and what all these mean.

Consider the following two ways in which the moving body can produce excitations:

1. deflecting a helium atom, or
2. creating a phonon (particle of wave motion).

The dispersion relation for an atom is

$$E = \frac{p^2}{2m}.$$

So

$$\frac{E}{p} = \frac{p}{2m}.$$

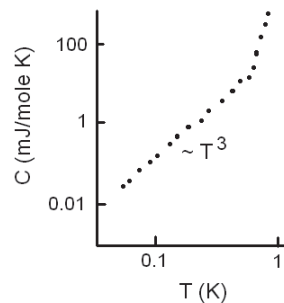
This depends of momentum  $p$  of the atom. So  $E/p$  could have a range of values. Since  $p$  can take any value from 0 upwards, so the range of possible  $E/p$  is also any value from 0 upwards.

Therefore, the smallest  $E/p$  is zero - when  $p = 0$ .

Therefore, the Landau critical velocity  $v_L = 0$ .

This means that there can be no superfluid in an ideal gas, which is made up of such free particles.

It is possible to verify that phonons are indeed the main contribution to heat capacity of liquid helium-4 at low temperature  $T$ . This was measured by D. S. Greywall in 1978. This picture is a sketch of his results.



The measured heat capacity varies as  $T^3$  below 1 K. This is consistent with the phonon contribution.

The dispersion relation for a phonon is

$$E = pc.$$

So

$$\frac{E}{p} = c.$$

This means that  $E/p$  is constant. There is only one possible for  $E/p$ , so the smallest  $E/p$  is also  $c$ .

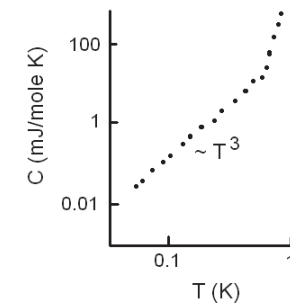
Therefore, the Landau critical velocity  $v_L = c$ .

This means that the fluid would be superfluid, as long as the body does not move faster than the speed of sound.

As long as the body moves slower than the speed of sound, it cannot lose energy to excite any phonon. So it would not experience any viscosity.

To see how this can help explain why helium-4 is a superfluid, lets pursue the phonon idea further.

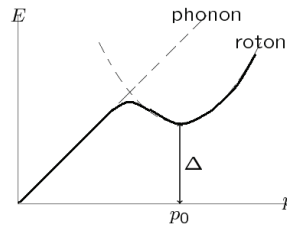
Above 1 K, the heat capacity increases faster than  $T^3$  - notice that the slope gets steeper.



In 1947, Landau, a Russian physicist, suggested that this is due to a new type of excitation that is different from phonon. He thought it has something to do with rotation of the liquid, so he called it "roton".

Landau suggested that, above 1 K, the change in heat capacity from  $T^3$  was due to an excitation different from phonon.

He assumed that it has a dispersion relation given by the parabolic curve labelled “roton” in this graph.



By choosing suitable height and width of this parabola, he could fit the measured heat capacity results.

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### Experiment.

The Landau critical velocity is then given approximately by

$$v_L = \frac{\Delta}{p_0} = 58 \text{ m s}^{-1}.$$

In 1977, Allum setup an experiment to accelerate a ball of ions through superfluid helium-4. They measured the resistance.

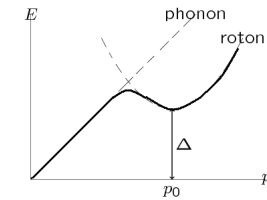
They found that the resistance remained zero until the velocity of the ball reached 45 m/s. Then the resistance increased rapidly.

45 m/s and 58 m/s are within the same order of magnitude. Given the complexity of the system, this is accepted as reasonable evidence that Landau's theories are correct.

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In 1961, Henshaw and Woods measured the dispersion relation of excitations in liquid helium-4 using neutron scattering. They confirmed that Landau's dispersion curve was correct.



Their measurements gave the following quantities:

$$\text{excitation: } \frac{\Delta}{k_B} = 8.65 \text{ K},$$

$$\text{momentum: } \frac{p_0}{\hbar} = 19.1 \text{ nm}^{-1}.$$

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### London's explanation of superfluidity

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Fritz London suggested using Bose-Einstein condensate (BEC) to explain superfluidity. This is possible because:

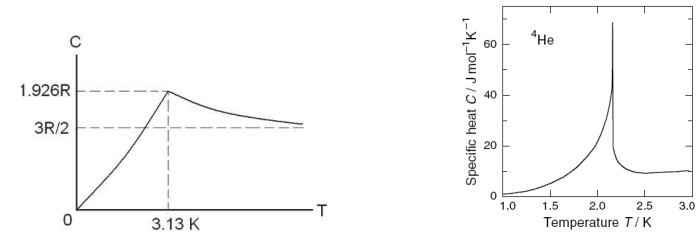
1. Helium-4 is a boson, so it may undergo Bose Einstein condensation.
2. The liquid would then become a single wavefunction, so it would show no viscosity unless it flows too quickly.

We shall look at a simplified version of London's treatment to calculate the BEC heat capacity.

This involves finding:

1. the temperature at which condensation starts,
2. the change in number of atoms with temperature, and
3. the heat capacity below condensation temperature.

In order to test his theory, London calculated the transition temperature and heat capacity of the BEC (left figure). He showed that there is some agreement with liquid helium-4 (right figure), at least in trend.



Left: Enss and Hunklinger, Low Temperature Physics, page 8, 2005.

Right: London, Physical Review, volume 54, page 947, 1938

Helium-4 is a boson. The energy distribution of bosons is given by the Bose-Einstein distribution:

$$n(\epsilon)d\epsilon = \frac{g(\epsilon)d\epsilon}{\exp((\epsilon - \mu)/k_B T) - 1}$$

Using this, we can calculate the temperature at which bosons condense into BEC.



We have seen a number of different density of states.

For electrons, we need to multiply by 2 because of the spin states. For phonons, we multiply by 3 for the three possible polarisations. For photons, we multiply by 2 for the two polarisation states.

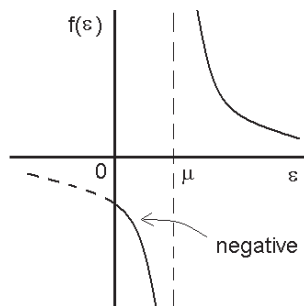
So what do we do for atoms that are bosons?

The answer is: we use the same density of state for the ideal gas:

$$g(\varepsilon) = \frac{4m\pi V}{h^3} (2m\varepsilon)^{1/2}$$

This is the very first one that we have seen, before we have to include the additional effects of spin and polarisation.

$$f(\varepsilon) = \frac{1}{\exp((\varepsilon - \mu)/k_B T) - 1}$$



If  $\mu$  is positive,  $f(\varepsilon)$  is negative for some energies. Then some states have a negative number of particles, which is not possible.

$\mu$  is called the chemical potential. It is determined by the total number of particles. We get this by integrating:

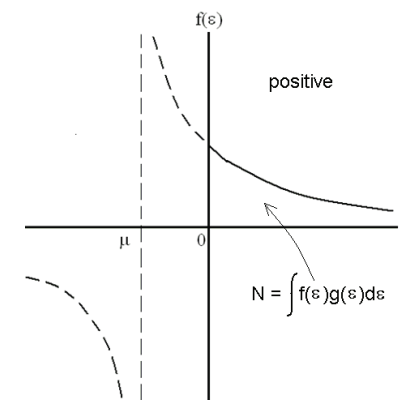
$$N = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp((\varepsilon - \mu)/k_B T) - 1}$$

I have assumed that the ground state energy is zero, which is why I integrated from zero.

This is not easy to solve. Instead, we shall make use of some approximations at low temperature.

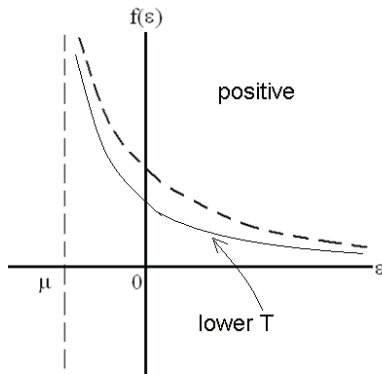
For a start, note that  $\mu$  could depend on temperature, since the above equation contains  $T$ .

So  $\mu$  must be negative.



The graph to the right of 0 contributes to the total particle number  $N$ .

If temperature falls,  $f(\epsilon) = \frac{1}{\exp((\epsilon - \mu)/k_B T) - 1}$  decreases.



Then particle number decreases, which is not possible.

We focus on the occupation number in the integral:

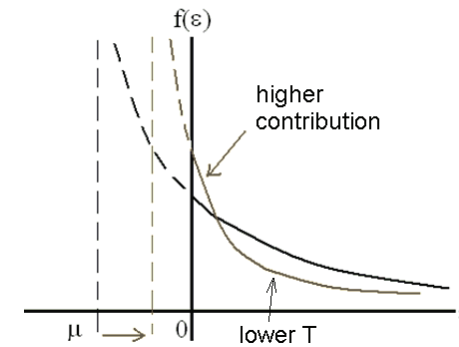
$$f(\epsilon) = \frac{1}{\exp((\epsilon - \mu)/k_B T) - 1}$$

First, note that  $\mu$  cannot be positive, or else for small energy the exponential function would be less than one. Then the denominator would be negative. This means negative occupation number, which is not physical.

Next, we know that when temperature is low enough, a large number of atoms would go into the ground state  $\epsilon = 0$ . The number would be as large as  $N$ , which is of the order of 1 mole ( $\approx 10^{24}$ ).

This means that  $f(0)$  would be very large, so that  $\mu$  must be very close to zero. Then the exponential function would be close to 1 and the denominator would be close to zero.

The particle number can remain fixed if  $\mu$  moves closer to 0:



Then  $f(\epsilon)$  increases and particle number can remain the same.

So at very low temperature, it is safe to assume that  $\mu = 0$ . We can then write

$$N_{ex} = \int_0^\infty \frac{g(\epsilon)d\epsilon}{\exp(\epsilon/k_B T) - 1}$$

I have added a subscript  $ex$  to  $N$ . The reason will become clear in a moment. First, integrate this with the help of the Table of Integrals:

$$N_{ex} = \left( \frac{2\pi m k_B T}{h^2} \right)^{3/2} 2.612V$$

Notice that  $T$  is in the denominator. This suggests that the total number of atoms decreases with temperature - that they are disappearing !

In fact, the above integral includes only atoms above the ground state  $\epsilon = 0$ . The "missing" atoms are going into the ground state.

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### The condensation temperature

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The formula for  $N_{ex}$  is only able to compute the number of particles above the ground state. This is because the density of states  $g(\varepsilon)$  in the integral is zero when  $\varepsilon = 0$ .

When a substantial fraction of the particles start going into the ground state, this has to be added separately. So  $N_{ex}$  is the number of atoms in the excited states. Hence the subscript *ex*.

We know that as soon as  $N_{ex}$  becomes less than the original number  $N$ , then  $(N - N_{ex})$  atoms start going into the ground state.

Therefore, condensation takes place when  $N_{ex} = N$ . Substituting this and solving for  $T$ , we get

$$T_{BE} = \frac{h^2}{2\pi m k_B} \left( \frac{N}{2.612V} \right)^{2/3}.$$

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### Heat capacity.

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Next, to calculate the heat capacity, recall the expression for the number of excited particles:

$$N_{ex} = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp(\varepsilon/k_B T) - 1}.$$

To find the heat capacity, we first find the energy  $U$ . The above integral is a sum over particle number in every energy interval.

To find  $U$ , we need to multiply by the energy  $\varepsilon$  at each interval  $d\varepsilon$ :

$$U = \int_0^\infty \frac{\varepsilon g(\varepsilon)d\varepsilon}{\exp(\varepsilon/k_B T) - 1}.$$

Integrating using the table of integrals gives

$$U = 0.7704 k_B N \frac{T^{5/2}}{T_{BE}^{3/2}}.$$

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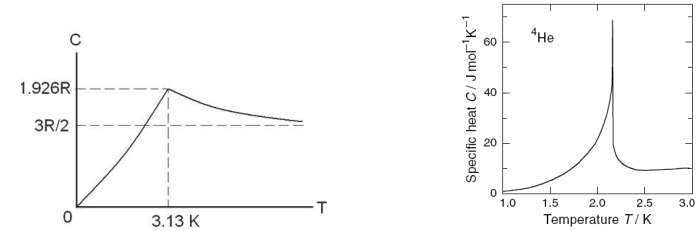
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### Transition temperature

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Substituting the mass of helium atom and the molar volume into the formula for the condensation temperature  $T_{BE}$ , we find 3.13 K.

This is fairly close to the lambda point temperature of 2.18 K, at which liquid helium-4 changes to a superfluid,



and provides some support for the idea that the superfluid is a BEC.

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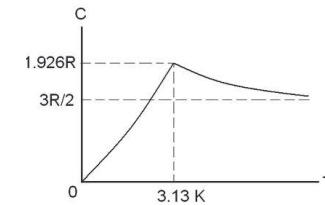
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### Below condensation temperature.

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The heat capacity below the condensation temperature is then obtained by differentiating with respect to  $T$ :

$$C = 1.926 k_B N \left( \frac{T}{T_{BE}} \right)^{3/2}.$$



Note that the peak value is  $1.926 k_B N$ . This is obtained by setting  $T = T_{BE}$ , when  $N_{ex} = N$  (all atoms are just excited).

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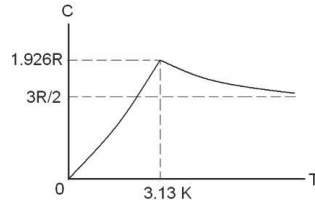
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Above condensation temperature.

Above condensation temperature  $T_{BE}$ , we cannot use the same formula for the heat capacity. This is because it is derived assuming that the chemical potential  $\mu \approx 0$ , which is only true when condensation starts.

Above  $T_{BE}$ ,  $\mu$  changes. Albert Einstein has published a formula for this in 1924 (right half of curve):



At high temperatures, the heat capacity reaches the ideal gas value of  $3Nk_B/2$ .

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Condensation Temperature

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I have said earlier that at very low temperature, we may set the chemical potential  $\mu$  to zero, so that the total number:

$$N = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp((\varepsilon - \mu)/k_B T) - 1},$$

becomes the excited number:

$$N_{ex} = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp(\varepsilon/k_B T) - 1},$$

I have not actually justified why it is valid to set  $\mu$  to zero. We shall do this now.

Recall the density of states formula

$$g(\varepsilon) = \frac{4m\pi V}{h^3} (2m\varepsilon)^{1/2}$$

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Condensation

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When  $\mu$  is set to zero, the integral becomes:

$$\int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp(\varepsilon/k_B T) - 1} = \frac{4m\pi V}{h^3} (2m)^{1/2} \int_0^\infty \frac{\varepsilon^{1/2} d\varepsilon}{\exp(\varepsilon/k_B T) - 1}.$$

This can be integrated using the result

$$\int_0^\infty \frac{x^{1/2}}{e^x - 1} dx = 2.315$$

to give the finite answer

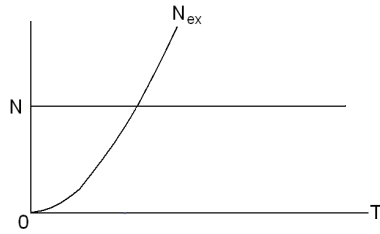
$$\int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp(\varepsilon/k_B T) - 1} = \left( \frac{2\pi m k_B T}{h^2} \right)^{3/2} 2.612V$$

Denote this by  $N_{ex}$  and sketch the graph against  $T$ .

Since  $N_{ex}$  is related to  $T^{3/2}$ , it increases from zero and eventually becomes larger than  $N$ .

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To understand what this means, consider the original integral

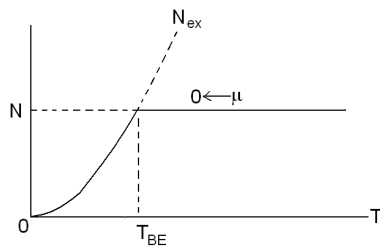
$$N = \int_0^{\infty} \frac{g(\varepsilon)d\varepsilon}{\exp((\varepsilon - \mu)/k_B T) - 1}.$$

$\mu$  can be adjusted so that the integral is equal to  $N$ . Assuming that this is the case, a graph of  $N$  versus  $T$  is just a horizontal line.

Consider a graph of the integral

$$\int_0^{\infty} \frac{g(\varepsilon)d\varepsilon}{\exp((\varepsilon - \mu)/k_B T) - 1}$$

against  $T$ .



As  $T$  falls,  $\mu$  can be adjusted for the integral to be equal to  $N$ . When  $T$  reaches  $T_{BE}$ ,  $\mu$  reaches its maximum value of 0.

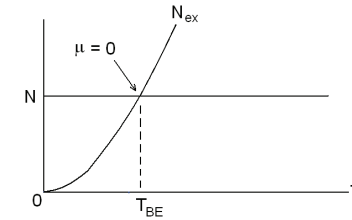
Compare

$$N_{ex} = \int_0^{\infty} \frac{g(\varepsilon)d\varepsilon}{\exp(\varepsilon/k_B T) - 1}$$

and

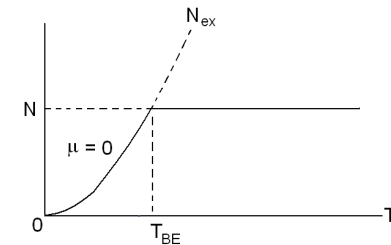
$$N = \int_0^{\infty} \frac{g(\varepsilon)d\varepsilon}{\exp((\varepsilon - \mu)/k_B T) - 1}.$$

They are equal when  $\mu = 0$ . When they are equal, the two graphs also meet.



Denote the temperature of the meeting point by  $T_{BE}$ .

Then  $\mu$  can only stay 0 because it cannot be positive. So the integral is now equal to  $N_{ex}$ . As  $T$  falls further, the integral falls along the  $N_{ex}$  curve.



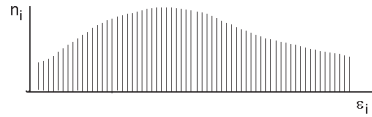
This also means that the integral

$$N = \int_0^{\infty} \frac{g(\varepsilon)d\varepsilon}{\exp((\varepsilon - \mu)/k_B T) - 1}.$$

fails for temperature below  $T_{BE}$  - it can no longer be equal to  $N$ .

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This is because  $g(\varepsilon)$  is only an approximate formula.



Recall from the ideal gas lectures that  $g(\varepsilon)$  is used when there are many energy levels for each energy interval  $d\varepsilon$ .

At very small  $T$ , only a few states near zero energy are populated, so  $g(\varepsilon)$  becomes inaccurate. In particular,  $g(0) = 0$  means that the integral  $\int f(\varepsilon)g(\varepsilon)d\varepsilon$  does not include the ground state ( $\varepsilon = 0$ ) contribution.

This is all right at room temperature where there are very few particles in the ground state. At very low temperatures when most of the particles go into the ground state, this is not valid any more.

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This understanding gives us a way to interpret the decreasing  $N_{ex}$ . It is the number of particles that have not fallen into the ground state yet - i.e. the number of excited particles. This means that  $T_{BE}$  is the temperature at which particles start going into the ground state at a macroscopic (very large) scale - it is the condensation temperature.

The number of excited particles is given by

$$N_{ex} = \left( \frac{2\pi m k_B T}{h^2} \right)^{3/2} 2.612V$$

So at  $T_{BE}$ ,  $N_{ex}$  is equal to  $N$ . Substituting and solving, we find

$$T_{BE} = \frac{h^2}{2\pi m k_B} \left( \frac{N}{2.612V} \right)^{2/3}.$$

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This means that if we want to calculate the total energy or number of particles, we should really add up the number in each energy level one by one, like this:

$$N = \sum n_i.$$

We cannot integrate using density of states anymore.

The idea of using  $N_{ex}$  is a compromise. We still calculate the number of particles in all energy levels above ground state by integrating using density of states. But we treat the ground state separately, and add  $N_{ex}$  to the number in the ground state to get the total:

$$N = N_0 + \int_0^\infty f(\varepsilon)g(\varepsilon)d\varepsilon,$$

where  $N_0$  is the number of particles in the ground state, and the integral is number ( $N_{ex}$ ) in all of the higher states.

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Learning Outcome: You should be able to

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Sketch and describe the phase diagram of liquid helium-4.

Derive the dispersion relations for a phonon and a real particle in empty space.

State the formula for Landau critical velocity. Describe what it means and the conditions where it is valid.

Describe Landau's dispersion relation for liquid helium. Explain the evidence from heat capacity measurement.

Describe London's explanation for superfluidity.

State the formula for Bose-Einstein distribution. Explain how chemical potential changes with temperature for bosons.

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Explain how Bose-Einstein distribution is used to understand Bose-Einstein condensation. Derive the formulae for number of excited bosons and condensation temperature.

Sketch and explain the graph of boson numbers against temperature.

Sketch and explain the graph for heat capacity of helium gas around the condensation temperature.

### Example 1

(i) What excitation is present in an ideal gas and why? Write down the dispersion relation, explaining the symbols used.

(ii) Determine algebraically the critical velocity of a body moving through an ideal gas.

(iii) Determine the critical velocity graphically. What does the result mean about the ideal gas?

### Worked Examples

### Solution

(i) Deflecting of free particles. Because particles in ideal gas do not interact with each other, and so behave as free particles.

Dispersion relation is

$$E = \frac{p^2}{2m},$$

where  $p$  is momentum and  $E$  is energy of each particle.

(ii) Critical velocity is the minimum  $E/p$ .

For the above dispersion relation,

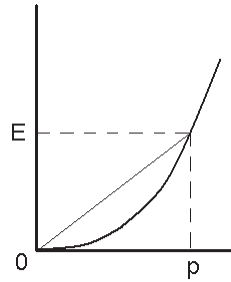
$$\frac{E}{p} = \frac{p}{2m},$$

The smallest  $p$  is 0. So minimum  $E/p$  is zero.

Therefore the critical velocity is zero.

(iii) First, sketch a graph of

$$E = \frac{p^2}{2m},$$



Then draw a straight line from origin to any point on the curve. The gradient of this line is  $E/p$ .

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### Example 2

(i) Sketch graphically the superfluid helium-4 dispersion relation. Label and describe the main features.

(ii) Sketch the graph of its heat capacity against temperature in logarithmic scales for both axes, for temperatures below superfluid transition.

(iii) Explain the connection between the two graphs.

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Move the point along the curve until this gradient is smallest.

In this case, the gradient is smallest when the point approaches the origin, where the straight line becomes horizontal.

Therefore the smallest gradient is then zero. So critical velocity is zero:

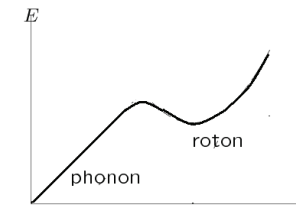
If a body does not move, there is no resistance. If it moves with any velocity, it will create excitations and experience resistance.

This means that the ideal gas is not a superfluid.

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### Solutions

(i)



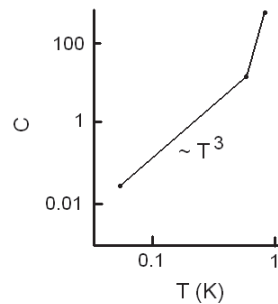
At low momentum  $p$ , the graph is straight, and only phonon excitations are possible.

At higher momentum, the graph bends to give a minimum. This comes from roton excitations

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(ii)



(iii) At low temperature, heat capacity is proportional to  $T^3$ . This is what phonons contribute, and suggests that only phonons are excited. This corresponds to the lower  $p$  region on the dispersion graph.

### Example 3

(i) Write down the integral expression for the total number of bosons in terms of density of states. Explain this integral.

(ii) The chemical potential increases to zero when temperature decreases to a certain value. What happens to the chemical potential and the integral expression when the temperature falls further?

(iii) How do we then interpret the integral expression? Why?

At higher temperature, the heat capacity deviates from  $T^3$ . This suggests that there is additional contributions from other types of excitations. This corresponds to the roton region on the dispersion graph.

### Solution

(i) The total number of bosons is given by

$$N = \int_0^\infty \frac{g(\epsilon)d\epsilon}{\exp((\epsilon - \mu)/k_B T) - 1}$$

$\epsilon$  is energy of a boson.  $g(\epsilon)$  is the density of states.

$g(\epsilon)d\epsilon$  is the number of states in energy interval  $d\epsilon$

$1/(\exp((\epsilon - \mu)/k_B T) - 1)$  is the average population of a state at energy  $\epsilon$ . When multiplied by  $g(\epsilon)d\epsilon$ , this gives the average population in  $d\epsilon$ .

Integrating then gives the total number of particles.

(ii) If we fix the chemical potential at zero, the resulting integral falls with temperature. So its value becomes smaller and smaller than the total number of particles.

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(iii) The original meaning of the integral as the total number of particles is not valid any more.

Instead, we can interpret the decrease of the integral with temperature as particles going into the ground state.

The reason is that  $g(\varepsilon)$  in the integral is zero at the ground state: The integral has not included the ground state in the first place.

The integral works at high temperature when the particles in the ground state are negligible. When a lot of particles go into the ground state at very low temperature, the integral is not accurate anymore.

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### Solutions

(i) The number of excited bosons below condensation temperature is

$$N_{ex} = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{\exp(\varepsilon/k_B T) - 1} = \frac{4m\pi V}{h^3} (2m)^{1/2} \int_0^\infty \frac{\varepsilon^{1/2} d\varepsilon}{\exp(\varepsilon/k_B T) - 1}.$$

(ii) The formula to use is

$$\int_0^\infty \frac{x^{1/2}}{e^x - 1} dx = 2.315$$

Comparing the denominator of the integrand, we make the substitution:

$$x = \frac{\varepsilon}{k_B T}.$$

Rearranging,

$$\varepsilon = k_B T x.$$

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### Example 4

(i) Write down the expression for the number of excited bosons at temperature below condensation.

(ii) Integrate this expression using the formula at the end of "Photon and phonons" lecture notes.

(iii) Derive the formula for the condensation temperature.

Since  $x$  is proportional to  $T$  the limits of 0 and  $\infty$  stay the same.

Substituting into the integral:

$$N_{ex} = \frac{4m\pi V}{h^3} (2m)^{1/2} \int_0^\infty \frac{(k_B T x)^{1/2}}{\exp(x) - 1} \times k_B T dx.$$

Regrouping,

$$N_{ex} = \left( \frac{2\pi m k_B T}{h^2} \right)^{3/2} V \times \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{x^{1/2}}{e^x - 1} dx.$$

Applying the formula,

$$\int_0^\infty \frac{x^{1/2}}{e^x - 1} dx = 2.315$$

we get

$$N_{ex} = \left( \frac{2\pi m k_B T}{h^2} \right)^{3/2} 2.612 V$$

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(iii) Condensation temperature  $T_{BE}$  is when the integral just starts to fall below  $N$ , the total number of particles.

So we can find this temperature by setting  $N_{ex}$  to  $N$ :

$$N = \left( \frac{2\pi m k_B T_{BE}}{h^2} \right)^{3/2} 2.612V.$$

Solving for  $T_{BE}$ , we get

$$T_{BE} = \frac{h^2}{2\pi m k_B} \left( \frac{N}{2.612V} \right)^{2/3}.$$